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B.Sc.– III

MATHEMATCS HONS. : Paper – V

group B. (Multiple integrals)

Contents :→ Volume integral, Gauss divergence theorem.

Volume Integral :→ Consider a continuous vector function $\vec{F}(\vec{R})$ & surface S enclosing the region E. Divide E into finite number of sub-regions $E_1, E_2, \dots, E_n$. Let δv_i be the volume of the subregion E_i enclosing any point whose position vector is $\vec{R}_i$

Consider the sum $V = \sum_{i=1}^{n} \vec{F}(\vec{R}_i)\, \delta v_i$

The limit of this sum as $n \to \infty$ in such a way $\delta v_i \to 0$, is called the volume integral of $\vec{F}(\vec{R})$ over E & is symbolically written as

$$\int_E \vec{F}\, dv$$

Remark :→ $\vec{F}(\vec{R}) = f(x,y,z)\,\hat{i} + \phi(x,y,z)\,\hat{j} + \psi(x,y,z)\,\hat{k}$

So that $\delta v = \delta x\, \delta y \cdot \delta z$, then

$$\int_E \vec{F}\, dv = \hat{i} \iiint_E f\, dx\,dy\,dz + \hat{j} \iiint_E \phi\, dx\,dy\,dz + \hat{k} \iiint_E \psi\, dx\,dy\,dz.$$

2.

Gauss Divergence theorem :→

(Relation between surface & volume integrals)

Statement :→ If $\vec{F}$ is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_E \operatorname{div}\vec{F}\,dv.$$

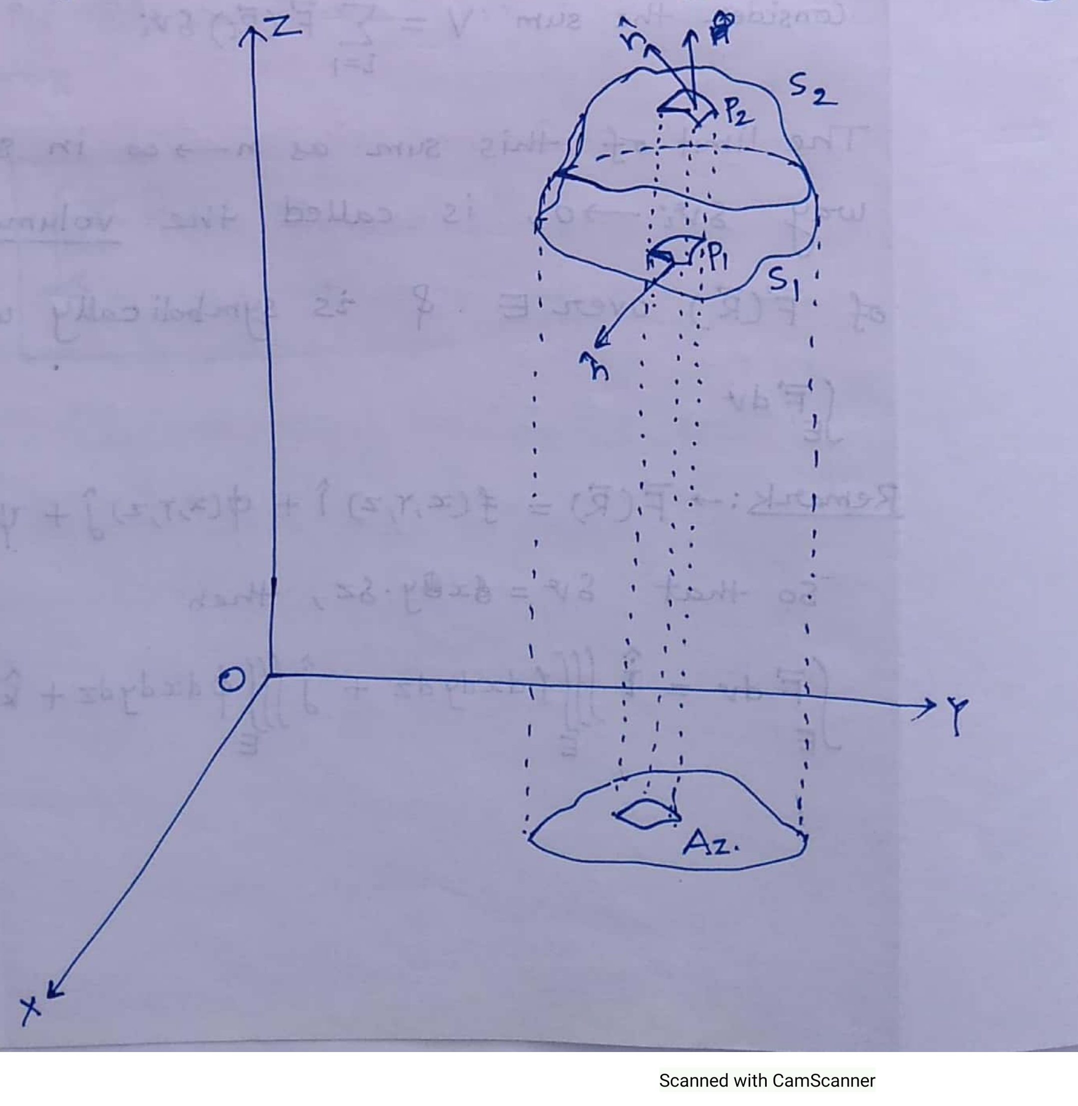
Proof :→

If $\vec{F}(\vec{R}) = f(x,y,z)\,\hat{i} + \phi(x,y,z)\,\hat{j} + \psi(x,y,z)\,\hat{k}$

Then it is required to prove that

$$\cancel{\iint_S f\,dy\,dz + \phi\,dy\,dz + \psi\,dx\,dy}$$

$$\iint_S f\,dy\,dz + \phi\,dz\,dx + \psi\,dx\,dy = \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz \quad \text{— (1)}$$



Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x,y,z_1)$ and $P_2(x,y,z_2)$ $(z_1 \leq z_2)$.

If S projects into the area A_z on the xy-plane, then

$$\iiint_E \frac{\partial \psi}{\partial z}\, dx\, dy\, dz = \iint_{A_z} dx\, dy \int_{z_1}^{z_2} \frac{\partial \psi}{\partial z}\, dz$$

$$= \iint_{A_z} \left[\psi(x,y,z_2) - \psi(x,y,z_1)\right] dx\, dy$$

$$= \iint_{A_z} \psi(x,y,z_2)\, dx\, dy - \iint_{A_z} \psi(x,y,z_1)\, dx\, dy \quad \text{—} (2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and $\hat{n}$ be the unit external normal vector at any point of S.

As the external normal at any point S_2 makes an ~~actu~~ acute angle with the positive direction of z-axis & that at any point of S_1 an obtuse angle, therefor.

$$\iint_{A_z} \psi(x,y,z_2)\, dx\, dy = \int_{S_2} \psi\, \hat{n}\cdot\hat{k}\, ds \quad \text{—} (3)$$

$$\& \iint_{A_z} \psi(x,y,z_1)\, dx\, dy = -\int_{S_1} \psi\, \hat{n}\cdot\hat{k}\, ds \quad \text{—} (4)$$

$\therefore$ from ③ & ④

$$\iiint_E \frac{\partial \psi}{\partial z}\, dx\, dy\, dz = \int_{S_2} \psi\, \hat{n}\cdot\hat{k}\, ds + \int_{S_1} \psi\, \hat{n}\cdot\hat{k}\, ds$$

$$\therefore \iiint_E \frac{\partial \psi}{\partial z}\, dx\, dy\, dz = \int_S \psi\, \hat{n}\cdot\hat{k}\, ds \quad \text{—— (5)}$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x}\, dx\, dy\, dz = \int_S f\, \hat{n}\cdot\hat{i}\, ds \quad \text{—— (6)}$$

$$\iiint_E \frac{\partial \phi}{\partial y}\, dx\, dy\, dz = \int_S \phi\, \hat{n}\cdot\hat{j}\, ds \quad \text{—— (7)}$$

Adding ⑤, ⑥ & ⑦, we have

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\, dy\, dz = \int_S (f\hat{i} + \phi\hat{j} + \psi\hat{k})\cdot\hat{n}\, ds$$

proved

$$\Rightarrow \boxed{\int_S \vec{F}\cdot\hat{n}\, ds = \int_E \operatorname{div}\vec{F}\, dv}$$

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